CORRIGENDUM TO MAHLER MEASURE OF A NON-RECIPROCAL FAMILY OF ELLIPTIC CURVES, Q. J. MATH. 74 (2023), 1187–1208

DETCHAT SAMART

There is a sign error in the formula in Theorem 1, which also has an impact on the formula for $\tilde{n}(\alpha)$ underneath. The corrected formulas are as follows:

Theorem 1. *Let $\tilde{n}(\alpha) = n(\alpha) - 3J(\alpha)$. For $\alpha \in (-1,3)\backslash\{0\}$, the following identities are true:*

$$\begin{aligned}\tilde{n}(\alpha) &= \frac{4}{1-3\operatorname{sgn}(\alpha)}\operatorname{Re}\left(\log\alpha - \frac{2}{\alpha^3}{}_4F_3\left(\begin{matrix}\frac{4}{3},\ \frac{5}{3},\ 1,\ 1\\ 2,\ 2,\ 2\end{matrix}\ \middle|\ \frac{27}{\alpha^3}\right)\right)\\ &= s(\alpha)\left(\frac{\sqrt[3]{2}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{2}\right)}{\sqrt{3}\pi^2}\alpha\,{}_3F_2\left(\begin{matrix}\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}\\ \frac{2}{3},\ \frac{4}{3}\end{matrix}\ \middle|\ \frac{\alpha^3}{27}\right) + \frac{\Gamma^3\left(\frac{2}{3}\right)}{2\pi^2}\alpha^2\,{}_3F_2\left(\begin{matrix}\frac{2}{3},\ \frac{2}{3},\ \frac{2}{3}\\ \frac{4}{3},\ \frac{5}{3}\end{matrix}\ \middle|\ \frac{\alpha^3}{27}\right)\right),\end{aligned}$$

where $s(\alpha) = -\frac{(1+3\operatorname{sgn}(\alpha))^2}{64}$.

Moreover, the equality

$$J(\alpha) = -\frac{1}{2\pi}\int_{\gamma_J'}\eta(x,y)$$

in the proof of Lemma 9 is incorrect. We need to use a different path with the prescribed endpoints to alter the integral $J(\alpha)$ so that its value remains unchanged. Below is a modified proof of Lemma 9.

Proof of Lemma 9. We label the six toric points as follows:

$$\begin{aligned}P_1^{\pm} &= (1, y_{\pm}(1)) = (1, Y_{\pm})\,,\\ P_2^{\pm} &= (e^{\pm ic(\alpha)}, y_{+}(e^{\pm ic(\alpha)})) = (Y_{\pm}, 1)\,,\\ P_3^{\pm} &= (e^{\pm ic(\alpha)}, y_{-}(e^{\pm ic(\alpha)})) = (Y_{\pm}, Y_{\pm})\,,\end{aligned}$$

where $c(\alpha) = \cos^{-1}\left(\frac{\alpha-1}{2}\right)$ and

$$Y_{\pm} = \frac{\alpha-1}{2} \pm \frac{\sqrt{(3-\alpha)(\alpha+1)}}{2}i.$$

Observe that $\tilde{n}(\alpha)$ can be rewritten as $\tilde{n}(\alpha) = I(\alpha) - 2J(\alpha)$, where

$$\begin{aligned}I(\alpha) &= \frac{1}{2\pi}\int_{-c(\alpha)}^{c(\alpha)}\log|y_{+}(e^{i\theta})|\mathrm{d}\theta,\\ J(\alpha) &= \frac{1}{2\pi}\int_{c(\alpha)}^{2\pi-c(\alpha)}\log|y_{+}(e^{i\theta})|\mathrm{d}\theta.\end{aligned}$$

Date: October 26, 2023.

Let $S = \{P_1^{\pm}, P_2^{\pm}, P_3^{\pm}\}$. Then we may identify the paths corresponding to $I(\alpha)$ and $J(\alpha)$ as elements in the relative homology $H_1(E_\alpha, S, \mathbb{Z})$, say γ_I and γ_J, respectively. In other words, we write

$$I(\alpha) = -\frac{1}{2\pi}\int_{\gamma_I} \eta(x,y), \quad J(\alpha) = -\frac{1}{2\pi}\int_{\gamma_J} \eta(x,y),$$

and boundaries of these paths can be seen as 0-cycles on S. Computing the limits of $y_+(e^{i\theta})$ as θ approaches $0, c(\alpha)$, and $-c(\alpha)$ from both sides, we find that

$$\lim_{\theta\to -c(\alpha)^+} y_+(e^{i\theta}) = \lim_{\theta\to c(\alpha)^-} y_+(e^{i\theta}) = 1,$$
$$\lim_{\theta\to 0^+} y_+(e^{i\theta}) = Y_-,$$
$$\lim_{\theta\to 0^-} y_+(e^{i\theta}) = Y_+.$$

Therefore, the path γ_I is discontinuous at $\theta = 0$ and

$$\partial\gamma_I = [[P_1^+] - [P_2^-]] + [[P_2^+] - [P_1^-]]. \tag{0.1}$$

This is illustrated in Figure 1 for $\alpha = 2$, where the dashed curves in the upper-half plane and the lower-half plane, both oriented counterclockwise, correspond to $\theta \in (-c(\alpha), 0)$ and $\theta \in (0, c(\alpha))$, respectively. Next, observe that

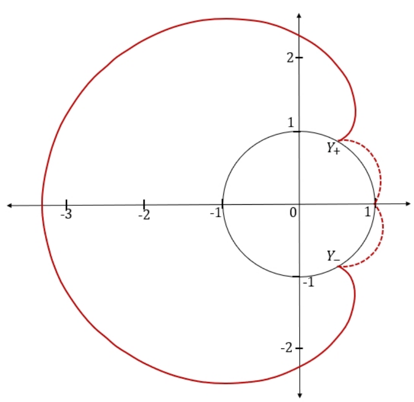

FIGURE 1. $y_+(e^{i\theta})$, $\theta \in [0, 2\pi)$

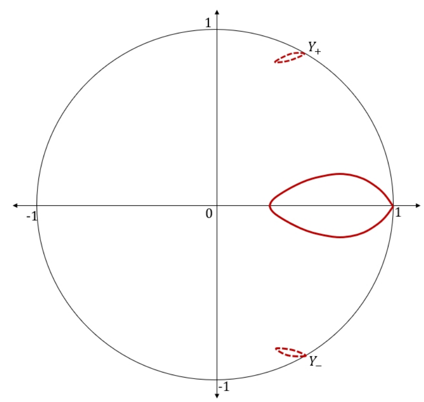

FIGURE 2. $y_-(e^{i\theta})$, $\theta \in [0, 2\pi)$

$$\lim_{\theta\to c(\alpha)^+} y_-(e^{i\theta}) = 1 = \lim_{\theta\to -c(\alpha)^-} y_-(e^{i\theta}),$$

and γ_J can be identified as the path $\{(e^{i\theta}, y_-(e^{i\theta})) \mid c(\alpha) < \theta < 2\pi - c(\alpha)\}$ (with reversed orientation), implying

$$\partial\gamma_J = [[P_2^+] - [P_2^-]]. \tag{0.2}$$

(For $\alpha = 2$, the y-coordinate of this path is the bold curve inside the unit circle, as illustrated in Figure 2, oriented clockwise.) Define

$$\gamma_J' = \left\{\left(\frac{1}{y_-(e^{i\theta})}, y_-\left(\frac{1}{y_-(e^{i\theta})}\right)\right) \mid c(\alpha) < \theta < 2\pi - c(\alpha)\right\}.$$

By some calculation, one sees that

$$y_-\left(\frac{1}{y_-\left(e^{i\theta}\right)}\right) = e^{-i\theta},$$
$$\lim_{\theta\to c(\alpha)^+} y_-(e^{i\theta}) = 1 = \lim_{\theta\to -c(\alpha)^-} y_-(e^{i\theta}),$$

implying

$$\partial\gamma'_J = [[P_1^+] - [P_1^-]]. \tag{0.3}$$

Moreover, we have

$$\begin{aligned}\int_{\gamma_J}\eta(x,y) &= -\int_{c(\alpha)}^{2\pi-c(\alpha)}\log\left|y_+\left(e^{i\theta}\right)\right|\mathrm{d}\theta\\ &= \int_{c(\alpha)}^{2\pi-c(\alpha)}\log\left|y_-\left(e^{i\theta}\right)\right|\mathrm{d}\theta\\ &= \int_{c(\alpha)}^{2\pi-c(\alpha)}\log(\left|1/y_-\left(e^{i\theta}\right)\right|)\mathrm{d}(-\theta)\\ &= \int_{\gamma'_J}\eta(x,y).\end{aligned}$$

Finally, we arrive at

$$\tilde{n}(\alpha) = I(\alpha) - 2J(\alpha) = -\frac{1}{2\pi}\left(\int_{\gamma_I}\eta(x,y) - \int_{\gamma_J}\eta(x,y) - \int_{\gamma'_J}\eta(x,y)\right) = -\frac{1}{2\pi}\int_{\tilde{\gamma}_\alpha}\eta(x,y),$$

where, by (0.1),(0.2), and (0.3), $\tilde{\gamma}_\alpha$ has trivial boundary, from which we can conclude that $\tilde{\gamma}_\alpha \in H_1(E_\alpha,\mathbb{Z})$. It is clear from the construction of the paths γ_I, γ_J, and γ'_J that they are anti-invariant under the action of complex conjugation. Therefore, we have $\tilde{\gamma}_\alpha \in H_1(E_\alpha,\mathbb{Z})^-$, as desired. □

I would like to thank Yusuke Nemoto and Zhengyu Tao for bringing these errors to my attention.

Department of Mathematics, Faculty of Science, Burapha University, Chonburi, Thailand 20131

Email address: petesamart@gmail.com